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Outline:

- More systems of ODEs
- When Ansätze go wrong

Last time:

- We classified 2D linear systems with constant coefficients
- Given $\dot{x} = Ax(t)$, $x(0) = x_0$, $A \in M_{n \times n}(\mathbb{R})$, $x \in \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
  $x(t) = \exp(tA)x_0$.
- Alternately if A has n linearly ind. eigenvectors $v_1, \dots, v_n$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$,

$$x(t) = c_1 v_1 e^{\lambda_1 t} + \cdots + c_n v_n e^{\lambda_n t}$$ for some constants $c_1, \dots, c_n$ determined by initial conditions.

This time:

How did we guess that $x(t) = c_1 v_1 e^{\lambda_1 t} + \cdots + c_n v_n e^{\lambda_n t}$?

Recall that $x(t) = \exp(tA)x_0 = P\exp(t\Lambda)P^{-1}x_0$ (if $A = P\Lambda P^{-1}$ an eigendecomposition)

$$= [v_1 \cdots v_n] \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \cdot P^{-1}x_0$$

Letting $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = P^{-1}x_0$,

$$x(t) = c_1 v_1 e^{\lambda_1 t} + \cdots + c_n v_n e^{\lambda_n t}.$$

Instead of finding $c_1, \dots, c_n$ by computing $P^{-1}x_0$, we can also just solve for $c_1, \dots, c_n$ by plugging $x(t) = c_1, \dots, c_n$ back into $\dot{x} = Ax$.

This is always true, but sometimes it'll give us a complex solution instead of a real one.

In the 2D case, where $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, we showed that

In the 2D case, where $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, we showed that a real solution is

$$x(t) = 2\cos(\beta t)e^{\alpha t} \cdot Re(v_1 y_{0,1}) - 2\sin(\beta t)e^{\alpha t} Im(v_1 y_{0,1}),$$

where $y_0 = P^{-1}x_0$, and $P = [v_1 \;\; v_2]$, the complex eigenvectors corresponding to λ_1, λ_2.

Computing this directly can involve lots of complex arithmetic to get back to a real answer. Can we make another choice?

Notice that $2 \cdot Re(v_1 y_{0,1})$ is a 2D real vector.
$-2 \cdot Im(v_1 y_{0,1})$ is a 2D real vector.

Thus, once we know $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$, we can guess the Ansatz

$$x(t) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{\alpha t}\cos\beta t + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} e^{\alpha t}\sin\beta t$$

and then plug that back into our original ODE to solve.

Ex. $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ $\quad \dot{x} = Ax$, $\quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\lambda^2 - 2\lambda + 5 = 0$$
$$\lambda = 1 \pm \sqrt{-4} = 1 \pm 2i$$

Ansatz: $x(t) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{t}\cos 2t + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} e^{t}\sin 2t$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{t}(-2\sin 2t) + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{t}\cos 2t + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} e^{t}(2\cos 2t) + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} e^{t}\sin 2t \\ Ax(t) = \begin{bmatrix} k_1 + 2k_2 \\ -2k_1 + k_2 \end{bmatrix} e^{t}\cos 2t + \begin{bmatrix} k_3 + 2k_4 \\ -2k_3 + k_4 \end{bmatrix} e^{t}\sin 2t \end{cases}$$

$$\begin{bmatrix} -2k_1 + k_3 \\ -2k_2 + k_4 \end{bmatrix}\sin 2t + \begin{bmatrix} k_1 + 2k_3 \\ k_2 + 2k_4 \end{bmatrix}\cos 2t = \begin{bmatrix} k_1 + 2k_2 \\ -2k_1 + k_2 \end{bmatrix}\cos 2t + \begin{bmatrix} k_3 + 2k_4 \\ -2k_3 + k_4 \end{bmatrix}\sin 2t$$

$$\Rightarrow \left.\begin{aligned} -2k_1 + k_3 &= k_3 + 2k_4 \\ -2k_2 + k_4 &= -2k_3 + k_4 \\ k_1 + 2k_3 &= k_1 + 2k_2 \\ k_2 + 2k_4 &= -2k_1 + k_2 \end{aligned}\right\} \qquad \begin{aligned} k_2 &= k_3 \\ k_1 &= -k_4 \end{aligned}$$

$$\Rightarrow \quad x(t) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{t}\cos 2t + \begin{bmatrix} k_2 \\ -k_1 \end{bmatrix} e^{t}\sin 2t$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} k_1 &= 1 \\ k_2 &= 1 \end{aligned}$$

$$x(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}\cos 2t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}\sin 2t$$

What happens if you guess wrong? Then either you won't end up with the right degrees of freedom (i.e. parameters) or the equations won't balance.

Ex. $\ddot{x} + 2\dot{x} + x = 0$

Guess $\quad x = A\sin t + B\cos t \qquad$ (bad guess)

$\quad \dot{x} = A\cos t - B\sin t$

$\quad \ddot{x} = -A\sin t - B\cos t$

$$\Rightarrow A\sin t + B\cos t + 2\dot{x} - A\sin t - B\cos t = 0$$

$$\Rightarrow 2\dot{x} = 0$$

$$\dot{x} = 0.$$

Note, this can be a solution. But only the trivial one of $x = 0$.
The parameters in our guess didn't play a role at all.

Guess $\quad \left.\begin{aligned} x &= At + B \\ \dot{x} &= A \\ \ddot{x} &= 0 \end{aligned}\right\} \quad \begin{aligned} &2A + At + B = 0 \\ &\Rightarrow A = 0, \quad B = 0 \quad \Rightarrow x = 0. \end{aligned}$

Again, this guess is bad.

Guess $\quad \left.\begin{aligned} x &= e^{At+B} \\ \dot{x} &= Ae^{At+B} \end{aligned}\right\} \quad \begin{aligned} &A^2 e^{At+B} + 2Ae^{At+B} + e^{At+B} = 0 \\ &A^2 + 2A + 1 = 0 \end{aligned}$

Guess $x = e^{At+B}$
$\dot{x} = Ae^{At+B}$
$\ddot{x} = A^2 e^{At+B}$

$$\left.\begin{aligned} x &= e^{At+B} \\ \dot{x} &= Ae^{At+B} \\ \ddot{x} &= A^2e^{At+B} \end{aligned}\right\} \quad \begin{aligned} &A^2e^{At+B} + 2Ae^{At+B} + e^{At+B} = 0 \\ &A^2 + 2A + 1 = 0 \\ &\Rightarrow A = -1 \end{aligned}$$

, B can be anything.

So $x = e^{-t+B}$ is a 1-parameter family of solutions

(Note, same as $x = Ce^{-t}$)

This is better, but we still suspect we ought to have a 2-parameter family.

If we were given IVP $x(0) = 1$, $\dot{x}(0) = 2$, this solution wouldn't help us.

Correct guess is of course $x = Ae^{-t} + Bte^{-t}$.

What about a particular solution to an inhomogeneous solution?

$$\ddot{x} + 2\dot{x} + x = t$$

Guess $x_p = Ae^t$. $\Rightarrow Ae^t + 2Ae^t + Ae^t = t \Rightarrow 4Ae^t = t$ ✗.

This time, the equations just didn't match at all.

$$\ddot{x} + 2\dot{x} + x = e^{-t}$$

Guess $x_p = Ae^{-t} \Rightarrow Ae^{-t} - 2Ae^{-t} + Ae^{-t} = e^{-t} \Rightarrow 0 = e^{-t}$ ✗

Again, things didn't work out.

Let $\dot{x} + x = e^{-t}$

$$\left.\begin{aligned} \text{Guess } x_p &= At^2e^{-t} \\ \dot{x}_p &= -At^2e^{-t} + 2Ate^{-t} \end{aligned}\right\} \quad \begin{aligned} At^2e^{-t} - At^2e^{-t} + 2Ate^{-t} &= e^{-t} \\ 2Ate^{-t} &= e^{-t} \\ 2At &= 1 \end{aligned}$$

✗

Guess $x_p = \underline{At^2e^{-t}} + \underline{Bte^{-t}} + \underline{Ce^{-t}}$

$\dot{x}_p = -\underline{At^2e^{-t}} + 2Ate^{-t} - \underline{Bte^{-t}} + Be^{-t} - \underline{Ce^{-t}}$

$\Rightarrow 2Ate^{-t} + Be^{-t} = e^{-t}$

$\Rightarrow A = 0,\ B = 1,\ C = ? \Rightarrow x_p = te^{-t} + Ce^{-t}$

$\Rightarrow 2Ae^{-t} + Be^{-t} = e^{-t}$

$\Rightarrow A = 0, \quad B = 1, \quad C = ? \Rightarrow \quad x_p = te^{-t} + Ce^{-t}$

If your Ansatz is too broad, it still can work out.

Teschl: 3.13 Let $A \in M_{n\times n}(\mathbb{R})$ and $\beta \in \mathbb{R}$.

Let $\dot{x} = Ax + p(t)e^{\beta t}$, where $p(t)$ is a vector all of whose entries are polynomials.

Let $\deg(p(t)) = \max\limits_{1 \le j \le n} \deg(p_j(t))$.

Then $x_p = q(t)e^{\beta t}$, for some polynomial vector $q(t)$ where $\deg(q(t)) = \deg(p(t))$ if β is not an eigenvalue of A, and $\deg(q(t)) = \deg(p(t)) + m$ if m is the multiplicity of the eigenvalue β.

Why can't we look at the degrees of each term separately?

Ex. $\dot{x} = Ax + p(t), \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad p(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$

Guess $x_p = \begin{bmatrix} k_1 t + k_2 \\ k_3 \end{bmatrix}$.

Then $\begin{bmatrix} k_1 \\ 0 \end{bmatrix} = \begin{bmatrix} k_3 \\ k_1 t + k_2 \end{bmatrix} + \begin{bmatrix} t \\ 1 \end{bmatrix}$

$\Rightarrow k_1 = k_3 + t$

$0 = k_1 t + k_2 + 1.$ ✗

Because the matrix can mix up the different entries.

So we need guess $x_p = \begin{bmatrix} k_1 t + k_2 \\ k_3 t + k_4 \end{bmatrix}$

Then $\begin{bmatrix} k_1 \\ \cdot \end{bmatrix} = \begin{bmatrix} k_3 t + k_4 \\ \cdot \end{bmatrix} + \begin{bmatrix} t \\ \cdot \end{bmatrix} \Rightarrow k_3 = -1, \quad k_1 = k_4$

Then $\begin{bmatrix} k_1 \\ k_3 \end{bmatrix} = \begin{bmatrix} k_3 t + k_4 \\ k_1 t + k_2 \end{bmatrix} + \begin{bmatrix} t \\ 1 \end{bmatrix} \Rightarrow$ $k_3 = -1, \quad k_1 = k_4$
$k_1 = 0, \quad k_2 = -2$

$\Rightarrow \quad x_p = \begin{bmatrix} -2 \\ -t \end{bmatrix}$